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# Holography in 4D (super) higher spin theories and a test via cubic scalar couplings

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Abstract: The correspondences proposed previously between higher spin gauge theories and free singleton field theories were recently extended into a more complete picture by Klebanov and Polyakov in the case of the minimal bosonic theory in $D = 4$ to include the strongly coupled fixed point of the 3d $O(N)$ vector model. Here we propose an $\mathcal{N} = 1$ supersymmetric version of this picture. We also elaborate on the role of parity in constraining the bulk interactions, and in distinguishing two minimal bosonic models obtained as two different consistent truncations of the minimal $N = 1$ model that retain the scalar or the pseudo-scalar field. We refer to these models as the type-A and type-B models, respectively, and conjecture that the latter is holographically dual to the 3d Gross-Neveu model. In the case of the type-A model, we show the vanishing of the three-scalar amplitude with regular boundary conditions. This agrees with the $O(N)$ vector model computation of Petkou, thereby providing a non-trivial test of the Klebanov-Polyakov conjecture.

Keywords: Field Theories in Lower Dimensions, Space-Time Symmetries, AdS-CFT and dS-CFT Correspondence, Supergravity Models.



 

## Contents



## 1. Introduction

A connection between three dimensional free field theory and massless higher spin (HS) theory in $AdS_4$ was proposed long ago in [1]. Motivated by the advances made in AdS/CFT correspondence in recent years, this connection was revisited in [2] and an increasingly sharpened picture of HS/CFT correspondence has begun to emerge [3–8]. The basic set up in this correspondence is as follows. Starting from a large number, $N$ say, of free fields and considering composite operators invariant under some flavor symmetry group, one identifies the generating functional with an effective action of a bulk theory with a perturbation series expansion in powers of $1/N$ around an AdS vacuum such that couplings of $m$ fields with $n$ derivatives are of order $L^n N^{-m/2}$ where $L$ is the AdS radius. This makes sense basically due to the fact that the CFT correlators factorize in the large-$N$ limit. The resulting bulk theory has two significant properties distinguishing it from the ordinary gauged supergravities. Firstly, the HS gauge theories have higher derivative interactions that are not small in units of the AdS radius, i.e. there is UV/IR mixing, while there is a weak field expansion scheme corresponding to the $1/N$-expansion [9]. Secondly, in addition to the stress-energy tensor the free field theory has flavor neutral conserved HS currents implying that the bulk theory has local HS symmetry in addition to local superdiffeomorphisms.

It is natural to first establish holography in this highly symmetric phase, and then break HS symmetry by introducing an additional mass-scale via some less symmetric solution thus showing holography in less symmetric phases of the theory. The solution may be of the form of a constant VEV for a dilaton-like scalar, as has been conjectured in the case of the $\mathcal{N} = 4$ theory in $D = 5$ [10, 7]. Here the additional parameter should be identified with the string tension/coupling $L^4 T_s^2 = N g_s$ in the type-IIB closed string theory on $\mathrm{AdS}_5 \times S^5$ of radius $L$ corresponding to finite Yang-Mills coupling $g^2_{\mathrm{YM}} = g_s$ on the field theory side. Another possibility is to consider domain-walls in which scalars with a non-zero mass-term are running. This has been proposed in the case of the $\mathcal{N} = 8$ theory in $D = 4$ [7] where the scalar should correspond to the 3d Yang-Mills coupling. Here the domain-wall is expected to interpolate between an unbroken phase of M theory on $\mathrm{AdS}_4 \times S^7$ and a broken phase described by ordinary supergravity.

In both of the above cases the flavor group is $\mathrm{SU}(N)$ and the unbroken bulk theory contains massless fields as well as massive fields (some of which are Goldstone modes) corresponding to bilinear and multi-linear single-trace operators, respectively. The symmetry breaking solutions discussed above involve massless as well as massive fields. Another possibility, equivalent to the proposal by Klebanov and Polyakov [8], is to consider domain wall solutions that involve only massless fields. We shall discuss this in more detail below after further remarks on the massless sector of the theory.

The underlying HS symmetry algebra is an infinite extension of the finite-dimensional AdS (super)group. Arranging the spectrum of composite operators into multiplets of the HS symmetry algebra there is a distinguished one formed out of the bilinears. This multiplet contains the stress-energy tensor, the HS currents as well as some additional lower spin operators. On the bulk side this set of operators corresponds to the quasi-adjoint, or twisted adjoint, representation of the HS algebra which is realized in the bulk as a zero-form master field that contains the lower spin fields, the spin $s \geq 1$ curvatures and their derivatives. In the free boundary field theory, the generating functional of $\mathrm{SU}(N)$ invariant composite operators can be consistently truncated to the one of the bilinear operators corresponding to the quasi-adjoint representation [7]. The corresponding bulk theory is an interacting and self-contained massless HS gauge theory.

The massless HS theories have been constructed by Vasiliev in $D = 4$ [11, 12] and in any $D$ in the case of no supersymmetry [13]. All of these theories have a minimal bosonic truncation consisting of massless fields with spin $s = 0, 2, 4, \dots$, corresponding to the a quadratic scalar operator (the mass-term), the stress-energy tensor and the spin $s = 4, 6, \dots$ currents of free scalars in $D - 1$ dimensions. The HS gauge theories in $D = 4$ with supersymmetry have been further elaborated upon in [14–16]. The interactions are constructed by introducing an adjoint one-form master gauge field and writing the field equations as constraints on the curvatures of the master fields. In fact, the most compact form of the constraints makes use of an internal non-commutative twistor-space and has a remarkably simple form in which all the curvatures carrying spacetime indices vanish.

Hence the local spacetime dependence can be gauged away completely [11, 12] which is the basic source for the UV/IR mixing. It is important to note, however, that the linearized theory has canonical kinetic terms, so that it is tachyon and ghost free, while higher derivatives enter only via interactions.

The above-mentioned three-dimensional SU($N$)-invariant scalar-singleton field theories are free in the UV, and in particular the sector of bilinear operators coincide with the free limit of the $O(N^2-1)$ vector model. The generating functional remains non-trivial, however, and it was conjectured that it corresponds holographically to the minimal bosonic HS gauge theory in $D=4$ based on the HS algebra $hs(4)$ [7]. This correspondence was then extended by Klebanov and Polyakov in [8] to include also the strongly coupled IR fixed point of the vector model, such that the two fixed points are the holographic duals of one and the same 4D bulk theory with two different scalar-field boundary conditions: the irregular boundary condition ($\Delta_-=1$) corresponds to the UV fixed point and the regular boundary condition ($\Delta_+=2$) corresponds to the IR fixed point. Remarkably, based on results on double-trace deformations [17 − 20], the Klebanov-Polyakov proposal implies that the resulting generating functionals are related to each other in the strict $N\to\infty$ limit via a simple Legendre transformation [17], which has been tested recently by Petkou [21]. This form of HS/CFT correspondence also contains an interesting alternative mechanism for breaking HS gauge symmetries by means of radiative corrections, in which the Goldstone modes are composite objects formed out of two massless fields [22]. This mechanism is available only in the case of regular boundary conditions, and the Goldstone modes correspond to the anomalous subleading $1/N$-corrections to the HS current conservation laws at the IR fixed point.

In this paper we elaborate on the role of parity in constraining the bulk interactions of the 4D HS gauge theories. In particular, we find two minimal bosonic models obtained as two different consistent truncations of the minimal $N=1$ model which we refer to as the type-A and type-B models, respectively. The $\mathcal{N}=1$ model contains a Wess-Zumino multiplet, and the type-A and -B models retain the scalar or the pseudo-scalar, respectively. We shall argue that the regular and irregular boundary conditions on the pseudo-scalar in type-B model yields the generating functional of 3d Gross-Neveu model expanded around the free and strongly coupled fixed points in the IR and UV, respectively. We also propose the holographic duals of the two generating functionals of the minimal $\mathcal{N}=1$ supersymmetric HS gauge theory associated with two distinct boundary conditions on the Wess-Zumino multiplet. These duals are the free and strongly coupled fixed points of an $\mathcal{N}=1$ supersymmetric $O(N)$ vector model.

As a non-trivial test of the holography in the type-A model, we show the vanishing of the three-scalar amplitude in the case that the scalar fields obey regular boundary conditions. This agrees with the result for the $O(N)$ vector model obtained long ago by Petkou [23].

There exists a generalization of the minimal HS gauge theories in $D=4$ that involves additional auxiliary fields [11]. The introduction of these fields does not change the cubic couplings of the physical fields. Hence the test of holography presented in this paper continues to hold in the generalized models as well.

| $(\ell, j)\backslash s$ | 0 | 1/2 | 1 | 3/2 | 2 | 5/2 | 3 | 7/2 | 4 | 9/2 | 5 | $\cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $(-1, \frac{1}{2})$ | $1 + \bar{1}$ | 1 | | | | | | | | | | |
| $(0, 0)$ | | | | 1 | 1 | | | | | | | |
| $(0, \frac{1}{2})$ | | | | | 1 | 1 | | | | | | |
| $(1, 0)$ | | | | | | | | 1 | 1 | | | |
| $(1, \frac{1}{2})$ | | | | | | | | | 1 | 1 | | |
| $\vdots$ | | | | | | | | | | | | |

**Table 1:** The spectrum of the minimal $\mathcal{N} = 1$ HS gauge theory arranged into levels labelled by $(\ell, j)$. The entries represent lowest weight representations $D(E_0, s)$ of SO$(3, 2)$ with ground states carrying AdS energy $E_0$ and SO$(3)$ spin $s$. Each level is an $\mathcal{N} = 1$ multiplet with $s_{\max} = 2\ell + 2 + j$. The AdS energies are given by $E_0 = s + 1$ for $s \geq 1/2$. The Wess-Zumino multiplet has two scalar lowest weight states with $E_0 = 1$ and $E_0 = 2$, which must be assigned even and odd parity, respectively, in the parity invariant $\mathcal{N} = 1$ model.

Further aspects of our results and the open problems will be discussed in the last section. Some of the results of section 4 overlap with those of [24] which appeared during the preparation of this paper.

## 2. Scalar couplings in the minimal $\mathcal{N} = 1$ model

In this section we give a brief description of the minimal $\mathcal{N} = 1$ HS model [15] and show the vanishing of the quadratic scalar self-couplings in the scalar field equation. We further demonstrate the vanishing of all higher order scalar contact terms in the scalar field equation, and discuss briefly the issue of field redefinitions.

### 2.1 The model

The minimal $\mathcal{N} = 1$ HS gauge theory is based on the HS algebra $hs(1|4)$, whose maximal finite-dimensional subalgebra is OSp$(1|4)$. The fundamental representation of $hs(1|4)$ is the OSp$(1|4)$ singleton $D(1/2, 0) \oplus D(1, 1/2)$. The massless spectrum of $hs(1|4)$ is the symmetric tensor product of two OSp$(1|4)$ singletons, and is given in table 1. Once the parity of the graviton is fixed to be even, the parities of all the other lowest weight states are fixed by HS symmetry since a generator carrying $n$ units of AdS energy ($n \in \frac{1}{2}\mathbf{Z}$) has parity $\exp i\pi n$. It follows that the parity of a lowest weight state with energy $E_0$ is given by $\exp i\pi(E_0 - 1)$. The sign in the parity assignments for the fermions is a matter of convention.

The basic building blocks of the theory are a master 0-form $\widehat{\Phi}$ and a master 1-form

$$\widehat{A} = dx^\mu \widehat{A}_\mu + dz^\alpha \widehat{A}_\alpha + d\bar{z}^{\dot{\alpha}} \widehat{A}_{\dot{\alpha}} \,, \tag{2.1}$$

where $x^\mu$ is the spacetime coordinate and $(z^\alpha, \bar{z}^{\dot{\alpha}})$ are Grassmann even SL$(2, \mathbf{C})$ spinor oscillators. The master fields are functions of $(x, z, \bar{z})$ as well as an additional set of internal oscillators $(y_\alpha, \bar{y}_{\dot{\alpha}}, \xi, \eta)$, where $(y_\alpha, \bar{y}_{\dot{\alpha}})$ are Grassmann even SL$(2, \mathbf{C})$ oscillators and $(\xi, \eta)$

are real, Grassmann odd oscillators. The Grassmann even oscillator algebra is represented by the following associative $\star$-products:[1]

$$\widehat{f}(z,\bar{z},y,\bar{y}) * \widehat{g}(z,\bar{z},y,\bar{y}) = \widehat{f}\exp\left[i\left(\frac{\overleftarrow{\partial}}{\partial y_\alpha}+\frac{\overleftarrow{\partial}}{\partial z_\alpha}\right)\left(\frac{\overrightarrow{\partial}}{\partial y^\alpha}-\frac{\overrightarrow{\partial}}{\partial z^\alpha}\right)+ + i\left(\frac{\overleftarrow{\partial}}{\partial \bar{y}_{\dot\alpha}}-\frac{\overleftarrow{\partial}}{\partial \bar{z}_{\dot\alpha}}\right)\left(\frac{\overrightarrow{\partial}}{\partial \bar{y}^{\dot\alpha}}+\frac{\overrightarrow{\partial}}{\partial \bar{z}^{\dot\alpha}}\right)\right]\widehat{g}\,. \tag{2.2}$$

The Grassmann odd oscillator algebra is given by

$$\xi\star\xi=1\,,\qquad \xi\star\eta=-\eta\star\xi=\xi\eta=-\eta\xi\,,\qquad \eta\star\eta=1\,. \tag{2.3}$$

At the kinematic level, the oscillator dependence of the master fields is restricted by the following conditions:

$$\tau(\widehat{A})=-\widehat{A}\,,\quad \widehat{A}^\dagger=-\widehat{A}\,,\qquad \tau(\widehat{\Phi})=\bar\pi(\hat\Phi)\,,\quad \hat\Phi^\dagger=\pi(\hat\Phi)\star\Gamma\,,\qquad \Gamma\equiv i\xi\eta\,, \tag{2.4}$$

where the $\star$-algebra anti-automorphism $\tau$ and automorphisms $\pi$ and $\bar\pi$ are defined by

$$\tau(\widehat{f}(z,\bar z,y,\bar y,\xi,\eta)) = \widehat{f}(-iz,-i\bar z,iy,i\bar y,i\xi,-i\eta)\,, \tag{2.5}$$
$$\pi(\widehat{f}(z,\bar z,y,\bar y,\xi,\eta)) = \widehat{f}(-z,\bar z,-y,\bar y,\xi,\eta)\,, \tag{2.6}$$
$$\bar\pi(\widehat{f}(z,\bar z,y,\bar y,\xi,\eta)) = \widehat{f}(z,-\bar z,y,-\bar y,\xi,\eta)\,. \tag{2.7}$$

The constraints giving rise to the full field equations are [15]

$$\widehat{F} = \frac{i}{4}dz^\alpha\wedge dz_\alpha\mathcal{V}(\widehat\Phi\star\kappa\Gamma)+\frac{i}{4}d\bar z^{\dot\alpha}\wedge d\bar z_{\dot\alpha}\bar{\mathcal{V}}(\widehat\Phi\star\bar\kappa)\,, \tag{2.8}$$
$$\widehat{D}\widehat{\Phi} = 0\,, \tag{2.9}$$

where the curvatures are defined as

$$\widehat{F} = d\widehat{A}+\widehat{A}\star\widehat{A}\,, \tag{2.10}$$
$$\widehat{D}\widehat{\Phi} = d\widehat{\Phi}+\widehat{A}\star\widehat{\Phi}-\widehat{\Phi}\star\bar\pi(\widehat{A})\,, \tag{2.11}$$

and the operators $\kappa$ and $\bar\kappa$ as

$$\kappa=\exp(iy^\alpha z_\alpha)\,,\qquad \bar\kappa=\kappa^\dagger=\exp(-i\bar y^{\dot\alpha}\bar z_{\dot\alpha})\,. \tag{2.12}$$

The quantity $\mathcal{V}(X)$ in (2.8) is an expansion in positive powers of $X$ using the $\star$-product, and $\bar{\mathcal{V}}(X)=(\mathcal{V}(X^\dagger))^\dagger$. Conventionally realized local Lorentz invariance requires [12]

$$\mathcal{V}(X)=b_1X+b_3X\star X\star X+\cdots\,, \tag{2.13}$$

where $b_1\neq 0$ in order for the theory to have well-defined linearized field equations. The master constraints (2.8) and (2.9) are invariant under the field redefinition $\hat\Phi\to F(\hat\Phi)$ where $F$ is a real and odd $\star$-function [2]. Using this freedom one can take

$$\mathcal{V}(X)=X\exp(i\theta(X))\,,\qquad \theta(X)=(\theta(X^\dagger))^\dagger=\theta_0+\theta_2X\star X+\cdots\,. \tag{2.14}$$

[1] This formula corrects a typo in [9].

The phase-factor $b_1 = \exp(i\theta_0)$ is inconsequential at the linearized level, where it can be absorbed into the Weyl tensors. Its first non-trivial appearance starts at the quadratic level in the field equations. The parameter $\theta_{2n}$ $(n = 1, 2, \ldots)$ appears for the first time in the $(2n+1)$th order in the field equations.

The parity map $P$ is defined by $P^2 = 1$ and

$$P(y_\alpha) = \bar{y}_{\dot\alpha}\,, \qquad P(z_\alpha) = -\bar{z}_{\dot\alpha}\,, \qquad P(\xi) = \xi\,, \quad P(\eta) = \eta\,, \tag{2.15}$$

from which it follows that $P$ is an automorphism of the $\star$-algebra, and that $P(\kappa) = \bar\kappa$, $P\pi = \bar\pi P$ and $P\tau = \tau P$. The parity transformations of the master fields are defined by

$$P(\widehat\Phi) = \widehat\Phi \star \Gamma\,, \qquad P(\widehat A) = \widehat A\,. \tag{2.16}$$

Parity invariance of the master constraints requires $\mathcal{V}(X)$ to be real which implies that $\theta(X) = 0$ in (2.14), that is[2]

$$\mathcal{V}(X) = X\,. \tag{2.17}$$

The parity of the physical fields following from (2.16) is discussed in the next section.

The master constraints are integrable, which ensures invariance under gauge transformations with $x$ and $Z$-dependent parameters. For the same reason, the spacetime field equations, which follow from

$$\widehat F_{\mu\nu}|_{Z=0} = \widehat D_\mu \widehat\Phi|_{Z=0} = 0\,, \tag{2.18}$$

are invariant under $x$-dependent gauge transformations, which incorporate spacetime diffeomorphisms as well as the local supersymmetry transformations. We note that while supersymmetry takes the standard form at the linearized level, it may be realized in an unconventional fashion at higher orders due to the particular nature of the higher derivative corrections to the constraints and transformation rules.

### 2.2 The scalar field equation up to second order in weak fields

In order to obtain the spacetime field equations from (2.18) one first uses the components of the master constraints (2.8) and (2.9) that carry at least one spinor index to solve for the $Z$-dependence of $\widehat\Phi$ and $\widehat A$ given an initial condition

$$A_\mu = \widehat A_\mu|_{Z=0}\,, \qquad \Phi = \widehat\Phi|_{Z=0}\,. \tag{2.19}$$

These fields have the following expansion in the $(y, \bar y, \xi, \eta)$-oscillators:

$$A_\mu = \sum_{\ell=0}^{\infty}\left(A_\mu^{(\ell,0)} + A_\mu^{(\ell,1/2)}\right), \tag{2.20}$$

$$\Phi = \Phi^{(-1,1/2)} + \sum_{\ell=0}^{\infty}\left(\Phi^{(\ell,0)} + \Phi^{(\ell,1/2)}\right), \tag{2.21}$$

[2]There are several other equivalent definitions of the parity map. For example, taking $P(\widehat\Phi) = -\widehat\Phi \star \Gamma$ implies that the parity invariant interactions are given by $\mathcal{V}(X) = iX$.

where $A_\mu^{(\ell,j)}$ and $\Phi^{(\ell,j)}$ are given by $(j=0,1/2)$

$$A_\mu^{(\ell,j)} = \sum_{m+n+p=4\ell+2+2j} A_{\mu,p}^{(\ell,j)}(m,n)\xi^p\eta^{2j}\,, \tag{2.22}$$

$$\Phi^{(\ell,j)} = C^{(\ell,j)} + \pi\Big((C^{(\ell,j)})^\dagger\Big)\star\Gamma\,, \tag{2.23}$$

$$C^{(\ell,j)} = \sum_{n-m-p=4\ell+2j+3} \Phi_p^{(\ell,j)}(m,n)\xi^p\eta^{1-2j}\,. \tag{2.24}$$

Here we use the short-hand notation

$$f(m,n) = \frac{1}{m!n!} y^{\alpha_1}\cdots y^{\alpha_m}\bar{y}^{\dot\alpha_1}\cdots\bar{y}^{\dot\alpha_n} f_{\alpha_1\dots\alpha_m\dot\alpha_1\dots\dot\alpha_n}\,. \tag{2.25}$$

The one-form $A_\mu$ contains the physical fields of the $\ell \geq 0$ multiplets in the spectrum listed in table 1. The $(-1,1/2)$ sector of the zero-form $\Phi$ contains the physical fields of the Wess-Zumino multiplet:

$$C^{(-1,1/2)} = \phi + \bar{y}^{\dot\alpha}\bar\lambda_{\dot\alpha}\xi + \cdots\,, \tag{2.26}$$

where the omitted terms are derivatives of the physical fields. From (2.16) it follows that the physical spin $s=2,4,\dots$ fields have even parity[3] and that

$$P(\phi) = \bar\phi\,, \tag{2.27}$$

which means that $\phi$ contains a scalar and a pseudo-scalar:

$$\phi = A + iB\,, \qquad P(A) = A\,, \qquad P(B) = -B\,. \tag{2.28}$$

The parity of the fermionic fields is a matter of convention, and depends on the choice of the phase-factors in the oscillator expansion (2.23) and (2.26).

The $Z$-dependence of $\widehat\Phi$ and $\widehat A$ can be obtained in a curvature expansion in powers of $\Phi$. The resulting form of the constraints (2.18) can be analyzed further in a modified expansion scheme in which both $\Phi$ and gauge fields residing in the $s_{\max} \geq 5/2$ multiplets (i.e. the multiplets labelled by $(0,1/2)$, $(1,0)$, $(1,1/2)$, $\dots$) are treated as weak fields. The resulting scalar field equation up to quadratic terms has been obtained for the minimal bosonic truncation of the $\mathcal{N}=1$ model in [9] (see Sect 3). The scalar field equation can be straightforwardly generalized to the $\mathcal{N}=1$ case, and the result is:

$$(D^\mu D_\mu + 2)\,\phi = \left(D^\mu P_\mu^{(2)} - \frac{i}{2}(\sigma^\mu)^{\alpha\dot\alpha}\frac{\partial}{\partial y^\alpha}\frac{\partial}{\partial\bar{y}^{\dot\alpha}}\,P_\mu^{(2)}\right)_{Y=\xi=\eta=0}\,, \tag{2.29}$$

where the supercovariant derivative is defined as

$$D_\mu\Phi = \nabla_\mu\Phi + \psi_\mu\star\Phi - \Phi\star\bar\pi(\psi_\mu)\,, \tag{2.30}$$

$$\psi_\mu = \frac{1}{2i}(\psi_\mu^\alpha Q_\alpha - \bar\psi_\mu^{\dot\alpha}\bar{Q}_{\dot\alpha})\,, \qquad Q_\alpha = y_\alpha\xi\,, \tag{2.31}$$

[3] The physical spin $s \geq 3/2$ fields arise in $(e^{-1})_a{}^\mu A_\mu$, which means that their parity transformation properties are composed from those of the vierbein and $A_\mu$. Note that $P(x^\mu) = x^\mu$, while $P$ acts non-trivially on local Lorentz indices. In particular, from (2.16) it follows that the anti-symmetric component of the vierbein $e_{\mu\,\alpha\dot\alpha}$ is odd under parity.

and

$$P^{(2)}_\mu = \Phi \star \bar\pi(W_\mu) - W_\mu \star \Phi + \\ + \left( \Phi \star \bar\pi(\widehat{E}_\mu{}^{(1)}) - \widehat{E}^{(1)}_\mu \star \Phi + \widehat{\Phi}^{(2)} \star \bar\pi(E_\mu) - E_\mu \star \widehat{\Phi}^{(2)} \right)_{Z=0}, \tag{2.32}$$

where $W_\mu$ contains the HS gauge fields, and we have defined $E_\mu = e_\mu + \psi_\mu$ and

$$\widehat{E}^{(1)}_\mu = -ie^{\alpha\dot\alpha}_\mu \int_0^1 \frac{dt}{t} \left( \left[ \bar y_{\dot\alpha}, \widehat{A}^{(1)}_\alpha \right]_* + \left[ \widehat{A}^{(1)}_{\dot\alpha}, y_\alpha \right]_* \right)_{z\to tz, \bar z \to t\bar z} + \psi_\mu\text{-terms}, \tag{2.33}$$

$$\widehat{\Phi}^{(2)} = z^\alpha \int_0^1 dt \left[ \Phi \star \bar\pi(\widehat{A}^{(1)}_\alpha) - \widehat{A}^{(1)}_\alpha \star \Phi \right]_{t\to tz, \bar z\to t\bar z} + \\ + \bar z^{\dot\alpha} \int_0^1 dt \left[ \Phi \star \pi(\widehat{A}^{(1)}_{\dot\alpha}) - \widehat{A}^{(1)}_{\dot\alpha} \star \Phi \right]_{t\to tz, \bar z \to t\bar z} \tag{2.34}$$

where

$$\widehat{A}^{(1)}_\alpha = -\frac{ib_1}{2} z_\alpha \int_0^1 tdt \Phi(-tz, \bar y, \xi, \eta) \kappa(tz, y) \star \Gamma, \tag{2.35}$$

$$\widehat{A}^{(1)}_{\dot\alpha} = -\frac{i\bar b_1}{2} \bar z_{\dot\alpha} \int_0^1 tdt \Phi(y, t\bar z, \xi, \eta) \bar\kappa(t\bar z, \bar y). \tag{2.36}$$

In (2.35), the quantity $\Phi(-tz, \bar y, \xi, \eta)$ is obtained from $\Phi(y, \bar y, \xi, \eta)$ simply by the substitution $y \to -tz$, and a similar operation is understood in (2.36).

### 2.3 Vanishing of quadratic scalar self-couplings

We are interested in the contribution to $P^{(2)}_\mu$ in (2.32) that are quadratic in the physical scalar $\phi$ and its derivatives. Denoting this contribution by $\widetilde{P}^{(2)}_\mu$ we write

$$P^{(2)}_\mu = \widetilde{P}^{(2)}_\mu + \cdots. \tag{2.37}$$

In computing $\widetilde{P}^{(2)}_\mu$, we first observe that the linearized Lorentz connection and auxiliary gauge fields in $W_\mu$ do not depend on the scalar field. Hence the Lorentz connection inside the kinetic term in (2.29) does not contribute to the quadratic self-couplings, nor does the $W_\mu$-terms in $P^{(2)}_\mu$. The remaining terms in $P^{(2)}_\mu$ are quadratic in $\Phi$, which means that we only need to keep track of contributions to $\Phi$ that are linear in $\phi$ and its derivatives. These are the components $\Phi^{(-1,1/2)}_0(m,m)$ $(m = 0,1,2,\ldots)$ occurring in (2.24), and which are given by the $m$th order derivative of $\phi$. The remaining components of $\Phi$ are either linear in physical fields other than $\phi$, or quadratic or higher order in physical fields. Thus

$$\widetilde{P}^{(2)}_\mu = \left( \Phi \star \bar\pi(\widehat{E}_\mu{}^{(1)}) - \widehat{E}^{(1)}_\mu \star \Phi + \widehat{\Phi}^{(2)} \star \bar\pi(E_\mu) - E_\mu \star \widehat{\Phi}^{(2)} \right)_{\substack{Z=0 \\ \Phi\to\Phi_0^{(-1,1/2)}}}. \tag{2.38}$$

A straightforward manipulation of the first two terms yields

$$\begin{aligned}\left(\Phi\star\bar\pi(\widehat{E}_\mu{}^{(1)})-\widehat{E}_\mu^{(1)}\star\Phi\right)_{\substack{Z=0\\ \Phi\to\Phi_0^{(-1,1/2)}}} &= ie_{\mu\,\gamma\dot\gamma}\sum_{m,n=0}^{\infty}\int_0^1\int_0^1 dt\;dt'\;t^n\;t'^{n+1}\frac{n}{(m!n!)^2}\times\\ &\quad\times\Phi^{(-1,1/2)}_{\alpha(m)\dot\alpha(m)}\star\Phi^{(-1,1/2)}_{\beta(n)\dot\beta(n)}\star \qquad (2.39)\\ &\quad\star\Big\{b_1\epsilon^{\dot\gamma\dot\beta_n}\Big[y^{\alpha(m)}\bar y^{\dot\alpha(m)},\; z^{\beta(n)\gamma}\bar y^{\dot\beta(n-1)}e^{itt'yz}\Big]_n\Gamma-\\ &\qquad-\bar b_1\epsilon^{\gamma\beta_n}\Big[\bar z^{\dot\beta(n)\dot\gamma}y^{\beta(n-1)}e^{-itt'\bar y\bar z},\; y^{\alpha(m)}\bar y^{\dot\alpha(m)}\Big]_n\Big\}_{Z=0},\end{aligned}$$

where $[A,B]_n\equiv A\star B+(-1)^nB\star A$, we use the shorthand notation $y^{\alpha(m)}=y^{\alpha_1}\cdots y^{\alpha_m}$, and

$$\Phi^{(-1,1/2)}_{\alpha(m)\dot\alpha(m)} = \Phi^{(-1,1/2)}_{0\,\alpha(m)\dot\alpha(m)}+(-1)^m\bar\Phi^{(-1,1/2)}_{0\,\alpha(m)\dot\alpha(m)}\Gamma\,, \qquad (2.40)$$

$$\Phi^{(-1,1/2)}_{0\,\alpha(m)\dot\alpha(m)} \sim i^m(\sigma^{\mu_1})_{\alpha_1\dot\alpha_1}\cdots(\sigma^{\mu_m})_{\alpha_m\dot\alpha_m}\Big(\nabla_{(\mu_1}\cdots\nabla_{\mu_m)}\phi-\text{traces}\Big)\,. \qquad (2.41)$$

Setting $Z=0$ in (2.39) enforces $m\geq n+1$, which in turn implies that at least $m-n+1$ $\bar y$-oscillators remains in the first term of (2.39). It follows that (2.39) contains at least two $y$ or $\bar y$-oscillators. Next we consider the second group of terms in (2.38), which can be written as

$$\begin{aligned}\left(\widehat\Phi^{(2)}\star\bar\pi(E_\mu)-E_\mu\star\widehat\Phi^{(2)}\right)_{\substack{Z=0\\ \Phi\to\Phi_0^{(-1,1/2)}}} &= \left(\frac{i}{2}e_{\mu\,\gamma\dot\gamma}\{\widehat\Phi^{(2)},y^\gamma\bar y^{\dot\gamma}\}_\star\right)_{\substack{Z=0\\ \Phi\to\Phi_0^{(-1,1/2)}}} \qquad (2.42)\\ &= B_\mu+\bar\pi\tau(B_\mu)+\pi[(B_\mu+\bar\pi\tau(B_\mu))^\dagger]\star\Gamma\,,\end{aligned}$$

where

$$\begin{aligned}B_\mu &= -\frac{b_1}{4}e_\mu^{\gamma\dot\gamma}\sum_{m,n=0}^{\infty}\int_0^1\int_0^1 dt\;dt'\;(t')^{n+1}\frac{1}{(m!n!)^2}\;\Phi^{(-1,1/2)}_{\alpha(m)\dot\alpha(m)}\star\Phi^{(-1,1/2)}_{\beta(n)\dot\beta(n)}\star\\ &\quad\star\Big\{z_\delta\Big[\Big(y^{\alpha(m)}\bar y^{\dot\alpha(m)}\Big)\star\Big(z^{\beta(n)\delta}\bar y^{\dot\beta(n)}e^{it'yz}\Big)\Big]_{z\to tz},\;y_\gamma\bar y_{\dot\gamma}\Big\}_\star\Gamma_{Z=0}\,. \qquad (2.43)\end{aligned}$$

Setting $Z=0$ forces $y_\gamma$ to contract $z_\delta$. It then follows that $\bar y_{\dot\gamma}$ cannot be contracted, since non-vanishing contributions to $\{A(Y,Z),B(Y,Z)\}_\star$ involve a total number of $YY$ and $ZZ$-contractions that is an even integer. Thus the number of $\bar y$-oscillators in $B_\mu$ is at least $|m-n|+1$. Contracting all the remaining $z$-oscillators inside the square-bracket requires $m\geq n+1$. Hence there are at least two $y$ or $\bar y$-oscillators in $B_\mu$.

We conclude that all contributions to $\tilde P^{(2)}_\mu$ contain at least two $y$ or $\bar y$-oscillators. Hence there are no contributions to the right hand side of the physical scalar field equation (2.29) that are quadratic in the physical scalar or its derivatives.

### 2.4 Vanishing of all non-derivative scalar self-couplings

In this section we strengthen the results of the previous section to show the vanishing of all non-derivative scalar self-couplings, i.e. the couplings depending only on the undifferentiated scalar. To show this it suffices to examine the scalar contact terms in the full scalar field equation which is given by

$$(D^\mu D_\mu + 2)\phi = \left( D^\mu P_\mu - \frac{i}{2}(\sigma^\mu)^{\alpha\dot\alpha} \frac{\partial}{\partial y^\alpha}\frac{\partial}{\partial \bar y^{\dot\alpha}}\, P_\mu \right)_{Y=\xi=\eta=0}, \tag{2.44}$$

where

$$\begin{aligned} P_\mu &= \Phi \star \bar\pi(W_\mu) - W_\mu \star \Phi + \\ &\quad + \sum_{n=2}^{\infty}\sum_{j=1}^{n} \left( \widehat\Phi^{(j)} \star \bar\pi(\widehat E_\mu^{(n-j)} + \widehat W_\mu^{(n-j)}) - (\widehat E_\mu^{(n-j)} + \widehat W_\mu^{(n-j)}) \star \widehat\Phi^{(j)} \right)_{Z=0}, \end{aligned} \tag{2.45}$$

and $\widehat E_\mu^{(n)}$ and $\widehat W^{(n)}$ are defined by

$$\widehat E_\mu = \frac{1}{1 + \widehat L^{(1)} + \widehat L^{(2)} + \cdots}\, E_\mu\,, \qquad \widehat W_\mu = \frac{1}{1 + \widehat L^{(1)} + \widehat L^{(2)} + \cdots}\, W_\mu\,, \tag{2.46}$$

where

$$\begin{aligned} \widehat L^{(n)}(\widehat f) = -i \int_0^1 \frac{dt}{t} \Bigg[ & \widehat A^{\alpha(n)} \star \left( \frac{\partial \widehat f}{\partial z^\alpha} - \frac{\partial \widehat f}{\partial y^\alpha} \right) + \left( \frac{\partial \widehat f}{\partial z^\alpha} + \frac{\partial \widehat f}{\partial y^\alpha} \right) \star \widehat A^{\alpha(n)} + \\ & + \widehat A^{\dot\alpha(n)} \star \left( \frac{\partial \widehat f}{\partial \bar z^{\dot\alpha}} + \frac{\partial \widehat f}{\partial \bar y^{\dot\alpha}} \right) + \left( \frac{\partial \widehat f}{\partial \bar z^{\dot\alpha}} - \frac{\partial \widehat f}{\partial \bar y^{\dot\alpha}} \right) \star \widehat A^{\dot\alpha(n)} \Bigg]_{z\to tz, \bar z \to t\bar z}. \end{aligned} \tag{2.47}$$

In what follows we use the notation

$$\widehat f|_\phi \tag{2.48}$$

to stand for the scalar contact terms in the quantity $\widehat f$. In order to obtain the scalar contact terms in (2.44) it suffices to compute the contributions to $P_\mu$ from $\widehat e_\mu|_\phi$. Higher order scalar contributions also enter (2.44) via the auxiliary gauge fields in $W_\mu$, but these are not of contact type. Recall that $\widehat A_\alpha$ and $\widehat\Phi$ are obtained from $\widehat F_{\alpha\dot\alpha} = \widehat D_\alpha \widehat\Phi = 0$ and $\widehat F^\alpha{}_\alpha = i\mathcal{V}(\widehat\Phi \star \kappa)$. We next observe that

$$\Phi|_\phi = \Phi_0^{(-1,1/2)}(0,0) = \phi + \bar\phi\Gamma\,. \tag{2.49}$$

The first terms in the $\Phi$-expansions of $\widehat A_\alpha$ and $\widehat\Phi$ are given in (2.34) and (2.35). From (2.35) it follows that

$$\widehat A_\alpha^{(1)}|_\phi = i z_\alpha(\phi + \bar\phi\Gamma) a^{(1)}(yz)\,, \qquad a^{(1)}(yz) = -b_1/2 \int_0^1 t dt \exp(ityz)\,. \tag{2.50}$$

Using $\bar\pi(\widehat A_\alpha^{(1)}|_\phi) = \widehat A_\alpha^{(1)}|_\phi$ it follows from (2.34) that

$$\widehat\Phi^{(2)}|_\phi = 0\,. \tag{2.51}$$

Iterating this procedure to all orders in the $\Phi$-expansion one finds that

$$\widehat\Phi|_\phi = \phi + \bar\phi\Gamma\,, \qquad \widehat A_\alpha|_\phi = i z_\alpha a(\phi + \bar\phi\Gamma; yz)\,, \tag{2.52}$$

where the function $a$ can be solved from the constraint on $\widehat{F}_{\alpha\beta}|_\phi$, which reads

$$\partial^\alpha(z_\alpha a) + i(z^\alpha a) \star (z_\alpha a) = \frac{1}{2}\mathcal{V}((\phi + \bar{\phi}\Gamma) \star \kappa\Gamma)\,. \tag{2.53}$$

Returning to (2.47), one sees that $\widehat{L}^{(n)}(e_\mu)|_\phi$ contains only contributions involving $e_\mu^{\alpha\dot{\alpha}}[\bar{y}_{\dot{\alpha}}, \widehat{A}_\alpha^{(n)}]_\star$, which vanish due to (2.52). Therefore, to all orders we find that

$$\widehat{e}_\mu|_\phi = e_\mu\,. \tag{2.54}$$

Hence, to all orders there is no contribution to $P_\mu$ that depends solely on $e_\mu$ and the undifferentiated scalar $\phi$. As a consequence the scalar field equation (2.44) does not contain any scalar contact interaction terms.

We conclude this section by discussing the role of field redefinitions in interpreting the results obtained above and in the previous section. To this end, we write the scalar contributions to the scalar field equation (2.44) as

$$\left(\nabla^2+\frac{2}{L^2}\right)\phi = \sum_{\substack{k\geq 2,\, n\geq 0\\ p_1+\cdots+p_k=2n}} L^{2n-2}\lambda^{\{\mu_1\ldots\mu_{p_1}\},\ldots,\{\nu_1\ldots\nu_{p_k}\}}(\nabla_{\mu_1}\cdot\cdot\nabla_{\mu_{p_1}}\phi)\cdots(\nabla_{\nu_1}\cdot\cdot\nabla_{\nu_{p_k}}\phi)\,, \tag{2.55}$$

where $L$ is the AdS radius and each group of indices is totally symmetric and traceless. The results of the two last sections imply that

$$\lambda^{\{\mu_1\ldots\mu_{p_1}\},\ldots,\{\nu_1\ldots\nu_{p_k}\}} = 0 \qquad \text{for } k=2 \qquad \text{or} \qquad p_1=\cdots=p_k=0\,. \tag{2.56}$$

An important property of the HS gauge theory is that the $\lambda$-coefficients are fixed numerical coefficients, i.e. they cannot be taken to be parametrically small. Hence all the higher derivative terms are of the same order and there is no sense in which one can take a low energy limit in which these terms become suppressed. For this reason the absence of the higher order contact interaction terms is less significant than it would be in an ordinary effective field theory in which there is a second length scale, such as the string length, above which the higher derivative interactions are energetically suppressed and one is left with only the contact terms.

The coefficients in (2.55) can be changed by redefining the scalar field as

$$\phi = \tilde{\phi} + \sum_{\substack{k\geq 2,\, n\geq 0\\ p_1+\cdots+p_k=2n}} \tau^{\{\mu_1\ldots\mu_{p_1},\ldots,\nu_1\ldots\nu_{p_k}\}}(\nabla_{\mu_1}\cdot\cdot\nabla_{\mu_{p_1}}\tilde{\phi})\cdots(\nabla_{\nu_1}\cdot\cdot\nabla_{\nu_{p_k}}\tilde{\phi})\,. \tag{2.57}$$

In particular, such redefinitions may yield couplings of the types excluded in (2.56). They do not, however, affect amplitudes, which is important for the test of holography presented in section 4.

Given that there are as many $\tau$-coefficients as $\lambda$-coefficients, it should be possible to eliminate all $\lambda$-coefficients with $n\geq 2$ order by order in $n$ for fixed $k$. The resulting contact terms and the mass-term together yield the scalar potential, while the $n=1$ coefficients define the Christoffel symbols on the sigma-model manifold.

In general, the physical field equations obtained from the master constraints contains arbitrarily high derivatives at any given non-linear order in the weak field expansion scheme. Whether field redefinitions can be used to obtain physical equations with only a finite number of derivatives at any given order in the weak expansion is not clear and requires further study.

## 3. Minimal bosonic type-A/B truncations of the $\mathcal{N}=1$ model

The minimal $\mathcal{N}=1$ model admits consistent bosonic truncations in which we retain $(\widehat{\Phi}_+, \widehat{A}_+)$ or $(\widehat{\Phi}_-, \widehat{A}_-)$ defined as

$$\widehat{\Phi}_\pm = i^{\frac{1}{2}(1\pm 1)}\frac{1}{2}(1\pm\Gamma)\star\widehat{\Phi}|_{\xi=\eta=0}\,, \qquad \widehat{A}_\pm = \frac{1}{2}(1\pm\Gamma)\star\widehat{A}|_{\xi=\eta=0}\,. \tag{3.1}$$

These master fields obey the $\tau$ and reality conditions (2.4) with $\Gamma$ set equal to 1 and the parity condition

$$P(\widehat{\Phi}_\pm) = \pm\widehat{\Phi}_\pm\,. \tag{3.2}$$

Actin with the $\pm$-projections on the $\mathcal{N}=1$ master constraints (2.8) and (2.9) we obtain

$$\widehat{F} = \frac{i}{4}dz^\alpha\wedge dz_\alpha\mathcal{V}_\pm(\widehat{\Phi}\star\kappa) + \frac{i}{4}d\bar{z}^{\dot\alpha}\wedge d\bar{z}_{\dot\alpha}\bar{\mathcal{V}}_\pm(\widehat{\Phi}\star\bar\kappa)\,, \tag{3.3}$$

$$\widehat{D}\widehat{\Phi} = 0\,, \tag{3.4}$$

where $\mathcal{V}_\pm$ are given in terms of the $\mathcal{V}$-function of the $\mathcal{N}=1$ model as

$$\mathcal{V}_+(X) = \mathcal{V}(X)\,, \qquad \mathcal{V}_-(X) = -\mathcal{V}(iX)\,. \tag{3.5}$$

We shall refer to the models keeping $(\widehat{\Phi}_+, \widehat{A}_+)$ and $(\widehat{\Phi}_-, \widehat{A}_-)$ as the type-A and type-B models, respectively. The spectrum of the two models are given by

$$\text{Type A}: \left[D\left(\frac{1}{2},0\right)\otimes D\left(\frac{1}{2},0\right)\right]_{\rm s} = D(1,0)\oplus D(3,2)\oplus D(5,4)\oplus\cdots\,, \tag{3.6}$$

$$\text{Type B}: \left[D\left(1,\frac{1}{2}\right)\otimes D\left(1,\frac{1}{2}\right)\right]_{\rm a} = D(2,0)\oplus D(3,2)\oplus D(5,4)\oplus\cdots\,, \tag{3.7}$$

where we recall that the parities are given by $(-1)^{E_0-1}$. From (2.28) and $\Phi_0^{(-1,1/2)}(0,0) = \phi+\bar\phi\Gamma$ it follows that the type-A model retains the scalar $A=\frac{1}{2}(\phi+\bar\phi)$, while the type-B model retains the pseudo-scalar $B=-\frac{1}{2}i(\phi-\bar\phi)$.

Starting from a parity invariant $\mathcal{N}=1$ model, in which case $\mathcal{V}(X)$ is given by (2.17), the resulting type-A and -B truncations remain parity invariant, and the corresponding $\mathcal{V}_\pm$-functions are given by

$$\text{Type A}: \ \mathcal{V}_+(X) = X\,, \tag{3.8}$$

$$\text{Type B}: \ \mathcal{V}_-(X) = -iX\,. \tag{3.9}$$



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## 4. Holography and a test via cubic scalar couplings

In this section we generalize the Klebanov-Polyakov conjecture [8] to the $\mathcal{N}=1$ model and the type-B model in the case of parity invariant interactions. We argue that the $\mathcal{N}=1$ model admit a family of boundary conditions dual to a line of fixed points of an $\mathcal{N}=1$ supersymmetric $O(N)$ vector model, which we examine in detail at the level of the scalar two-point function. We also propose that the type-B model is the AdS dual of the 3d Gross-Neveu model. Finally, the Klebanov-Polyakov conjecture is verified at the level of cubic scalar amplitudes.

### 4.1 The $\mathcal{N}=1$ model as AdS dual of super $O(N)$ vector model

In the case of parity invariant interactions the only free parameter of the minimal $\mathcal{N}=1$ model is the normalization of the action,[4] which we can take to be

$$S_{\rm cl} = \frac{N}{L^2}\int d^4x\mathcal{L}\,, \tag{4.1}$$

where $N$ is related to the number of free fields in the dual CFT, and the lagrangian $\mathcal{L}$ contains no additional free parameter. The $1/\sqrt{N}$ expansion of the action, which is obtained by rescaling the fields by $1/\sqrt{N}$, is equivalent to the weak field expansion scheme discussed in section 2. To compute amplitudes one needs to fix boundary conditions. OSp(1|4) symmetry in itself admits a one-parameter family of boundary conditions for the Wess-Zumino $(-1,1/2)$-multiplet. To describe these, we write the boundary behavior of $\phi = A+iB$ as

$$A = r\alpha_+ + r^2\beta_+\,,\qquad B = r\alpha_- + r^2\beta_-\,, \tag{4.2}$$

and define the $d=3$, $\mathcal{N}=1$ superfields (our conventions are given in the the appendix):[5]

$$\Phi_- = \alpha_- + i\bar\theta\eta_- + \frac{\bar\theta\theta}{2i}\beta_+\,, \tag{4.3}$$

$$\Phi_+ = \alpha_+ + i\bar\theta\eta_+ + \frac{\bar\theta\theta}{2i}\beta_-\,, \tag{4.4}$$

where the parities are given by $P(\alpha_\pm)=\pm\alpha_\pm$ and $P(\beta_\pm)=\pm\beta_\pm$, and the scaling dimensions by $\Delta(\alpha_\pm)=1$, $\Delta(\beta_\pm)=2$ and $\Delta(\eta_\pm)=3/2$. The OSp(1|4)-invariant boundary conditions are given by

$$B_\lambda:\quad \Phi_- - \lambda\Phi_+ = J\,, \tag{4.5}$$

where $J$ is a source field and $\lambda$ is an arbitrary real parameter which can be taken to be positive without any loss of generality. The resulting generating functionals for connected amplitudes are defined by

$$e^{iW_\lambda[J]} = e^{iS_{\rm cl}[\Phi_\lambda(J)]}\,, \tag{4.6}$$

where $\Phi_\lambda(J)$ denotes the solution to the bulk field equation subject to the $B_\lambda$ boundary

[4]Strictly speaking, at present only the full field equations are known. This, however, does not create an obstacle to the computation of cubic scalar amplitudes.

[5]In what follows we use a simplified notation in which the dependence on the $\ell\geq 0$ multiplets is suppressed.



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condition. Parity is broken for $\lambda \neq 0$, while it is restored at $\lambda = 0$ and $\lambda = \infty$, where the source fields are $\Phi_-$ and $\Phi_+$, respectively.

The $\mathcal{N} = 1$ supersymmetric extension of the arguments given by Klebanov and Witten in [17], suggests that $\Phi_-$ and $\Phi_+$ are conjugate variables in the $B_0$ theory in the sense that

$$\frac{\delta W_0}{\delta \Phi_-} = -\Phi_+ \,, \tag{4.7}$$

while the opposite should hold in the $B_\infty$ theory. This strongly suggests that the generating functionals $W_0[\Phi_-]$ and $W_\infty[\Phi_+]$ are related by the following parity and OSp(1|4) invariant Legendre transformation

$$W_\infty[\Phi_+] = W_0[\Phi_-] + \int d^3x d^2\theta \Phi_- \Phi_+ \,. \tag{4.8}$$

The scalar contributions to the second term are given by $\int d^3x(\beta_+\alpha_+ + \alpha_-\beta_-)$. The relative plus-sign is important, since $\Phi_\pm$ contains sources for mixed regular and irregular boundary conditions. The general argument given in [17], shows that a scalar field $S$ with kinetic term $-\frac{1}{2}\epsilon_S \int d^{d+1}x\sqrt{g}((\partial S)^2 + m^2 S^2)$, behaving near the boundary as $S \sim r^{d-\Delta} S_0 + r^\Delta \widetilde{S}_0$, yields generating functionals $W[S_0]$ and $\widetilde{W}[\widetilde{S}_0]$ for correlators of operators $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ of dimension $\Delta$ and $\widetilde{\Delta} = d - \Delta$, respectively, related by $\widetilde{W}[\widetilde{S}_0] = W[S_0] - \epsilon_S(2\Delta - d)\int d^dx S_0\widetilde{S}_0$. This leads to the rather puzzling prediction that $\epsilon_A = -\epsilon_B$. Remarkably enough, this is in perfect agreement with the result found in [28] for the bulk-stress tensor, which we have summarized in (5.5).

Turning to the holographic description of the $\mathcal{N} = 1$ HS gauge theory, we begin by observing that the $hs(1|4)$ symmetry requires the $B_0$ boundary condition. The $hs(1|4)$-invariant spectrum matches the bilinear operator content of a free $d = 3$, $\mathcal{N} = 1$ singleton field theory [7]. Moreover, rescaling the bulk fields by $1/\sqrt{N}$ leads to classical $n$-point amplitudes proportional to $(1/\sqrt{N})^{n-2}$. It is therefore natural to identify the generating functional of the classical $\mathcal{N} = 1$ HS gauge theory subject to the $B_0$ boundary condition with that of the free $\mathcal{N} = 1$ supersymmetric $O(N)$ model [7]:

$$e^{iW_0} = \left\langle \exp \sum_{(\ell,j)} \int d^3x d^2\theta \mathcal{O}_{(\ell,j)} \Phi_-^{(\ell,j)} \right\rangle_0 \,, \tag{4.9}$$

where $\mathcal{O}_{(\ell,j)}$ and $\Phi_-^{(\ell,j)}$ denote the bilinear operator and the corresponding source superfield of the level $(\ell, j)$ supermultiplet. In particular, the scalar operator coupling to the Wess-Zumino multiplet is given by:

$$\begin{aligned} \mathcal{O} &= \frac{c_1}{\sqrt{N}} W^2 = \mathcal{O}_1 + i\bar{\theta}\mathcal{O}_{3/2} + \frac{\bar{\theta}\theta}{2i}\mathcal{O}_2 \\ &= \frac{c_1}{\sqrt{N}}(\varphi\cdot\varphi + 2i\bar{\theta}\psi\cdot\varphi + \frac{\bar{\theta}\theta}{2i}(i\bar{\psi}\cdot\psi + 2f\cdot\varphi)) \,, \end{aligned} \tag{4.10}$$

where $W$ is the off-shell OSp(1|4) singleton superfield in the $O(N)$ vector representation, with the $\theta$-expansion

$$W = \varphi + i\bar{\theta}\psi + \frac{\bar{\theta}\theta}{2i} f \,, \tag{4.11}$$

and the constant $c_1 = 2^{3/2}\pi$ is chosen such that $\langle \mathcal{O}\mathcal{O} \rangle$ has unit strength.

The $\mathcal{N}=1$ supersymmetric extension of the arguments given by Witten [18] suggests that the $B_\lambda$ boundary condition corresponds to adding the double-trace deformation $\frac{\lambda}{2}\int \mathcal{O}^2$ to the free action:[6]

$$e^{iW_\lambda} = \left\langle \exp \sum_{(\ell,j)} \int d^3x d^2\theta \mathcal{O}_{(\ell,j)} \Phi_-^{(\ell,j)} \right\rangle_\lambda , \tag{4.12}$$

where $\langle \cdots \rangle_\lambda$ is evaluated using the action

$$\begin{aligned} S_\lambda &= \frac{i}{4}\int d^3x d^2\theta D^\alpha W D_\alpha W + \frac{\lambda}{2}\int d^3x d^2\theta \mathcal{O}^2 \\ &= \int d^3x \left( -\frac{1}{2}(\partial\varphi)^2 - \frac{i}{2}\bar{\psi}\sigma^\mu\partial_\mu\psi + \frac{1}{2}f^2 + \frac{\lambda}{2}(2\mathcal{O}_1\mathcal{O}_2 + i\bar{\mathcal{O}}_{3/2}\mathcal{O}_{3/2}) \right) . \end{aligned} \tag{4.13}$$

The $\lambda$-parameter breaks parity as well as HS symmetry, in agreement with the bulk side of the correspondence. It is known that (4.13) defines a superconformally invariant field theory for all values of $\lambda$. The absence of renormalizations of $\lambda$ can be demonstrated by examining the short-distance behavior of the $\exp(-\frac{\lambda}{2}\int d^3x d^2\theta \mathcal{O}^2)$ insertion. The factorization of correlators in the large-$N$ limit implies that

$$\begin{aligned} \left\langle \frac{1}{2}\left(\frac{i\lambda}{2}\int \mathcal{O}^2\right)^2 \right\rangle_0 &\simeq -\frac{\lambda}{2}\int dZ dZ' \mathcal{O}(Z) G(Z,Z') \mathcal{O}(Z') \\ &= -\frac{\lambda}{2}\int dZ \mathcal{O}(Z) \int d^3x' \frac{1}{|x'|^2} D^2 \mathcal{O}(Z+Z') . \end{aligned} \tag{4.14}$$

The deformation is well-behaved at short distances, and cannot produce any infinite nor finite corrections to $\lambda$, which shows the vanishing of the $\beta$-function at order $\lambda^2$. Next, let us examine the $\lambda$-dependence of the generating functional by studying the power-law behavior of the two-point function $\langle \mathcal{O}(Z)\mathcal{O}(Z')\rangle_\lambda$. Following Gubser and Klebanov [20], we rewrite the partition function

$$e^{iW_\lambda[J]} \equiv \left\langle \exp i \int \left( \frac{\lambda}{2}\mathcal{O}^2 + J\mathcal{O} \right) \right\rangle_0 , \tag{4.15}$$

by introducing an auxiliary superfield $\Sigma$, with the following result

$$e^{iW_\lambda[J]} = \sqrt{\det \frac{i}{\lambda}} \int D\Sigma \left\langle \exp i \int (-\frac{1}{2\lambda}\Sigma^2 + (\Sigma + J)\mathcal{O}) \right\rangle_0 . \tag{4.16}$$

Using the fact that the higher point functions of $\mathcal{O}$ are suppressed in the $1/N$ expansion, we approximate

$$\left\langle \exp i \int (\Sigma + J)\mathcal{O} \right\rangle_0 \simeq \exp -\frac{1}{2} \left\langle \left( \int (\Sigma + J)\mathcal{O} \right)^2 \right\rangle_0 , \tag{4.17}$$

[6] This deformation is the supersymmetric completion of the $\lambda \int d^3x \mathcal{O}_1\mathcal{O}_2$ deformation which was argued to be exactly marginal in [18]. As we shall discuss in section 5, the $hs(1|4)$ theory is not invariant under exchange of $A$ and $B$, which means that there is no invariance under $\lambda \to 1/\lambda$ of the type considered in [18].

With the help of this formula, the integral in (4.16) becomes a gaussian and it is readily evaluated to give

$$e^{iW_\lambda[J]} = \frac{1}{\sqrt{\det(1-i\lambda\widehat{G})}} \exp -\frac{1}{2}\int dZdZ'J(Z)\left[\frac{\widehat{G}}{1-i\lambda\widehat{G}}\right](Z,Z')J(Z')\,, \tag{4.18}$$

where

$$(\widehat{G}F)(Z) = \int dZ'G(Z,Z')F(Z')\,. \tag{4.19}$$

From (4.18) it follows that

$$\left\langle \mathcal{O}(Z)\mathcal{O}(Z')\right\rangle_\lambda = \left[\frac{\widehat{G}}{1-i\lambda\widehat{G}}\right](Z,Z')\,. \tag{4.20}$$

In superspace momentum basis we find

$$\begin{aligned}(\widehat{G}F)(P) &= \int \frac{dZ}{(2\pi)^{3/2}}\,e^{-iPZ}\,dZ'\,\frac{dP'}{(2\pi)^{3/2}}\,e^{iP(Z-Z')}\,\delta^2(\pi')\,G(p')\,\frac{dP''}{(2\pi)^{3/2}}\,F(P'')\\ &= -(2\pi)^{3/2}p^2G(p)\widehat{\gamma}(F(p,\pi))\,,\end{aligned} \tag{4.21}$$

where $G(p)$ is given in (A.11) and we have defined

$$\widehat{\gamma}(F(p,\pi)) = \int d^2\pi'\exp\Big(-\frac{\bar{\pi}\sigma^\mu p_\mu\pi'}{p^2}\Big)F(p,\pi')\,. \tag{4.22}$$

Using

$$\widehat{\gamma}^2(F(p,\pi)) = -\frac{1}{p^2}F(p,\pi) \tag{4.23}$$

we find that

$$\frac{\widehat{G}}{1-i\lambda\widehat{G}} = \frac{1}{1+4\pi^4\lambda^2}(\widehat{G}+i4\pi^4\lambda)\,, \tag{4.24}$$

from which it follows that

$$\left\langle \mathcal{O}(Z)\mathcal{O}(Z')\right\rangle_\lambda = \frac{1}{1+4\pi^4\lambda^2}\left\langle \mathcal{O}(Z)\mathcal{O}(Z')\right\rangle_0 + \text{contact-term}\,. \tag{4.25}$$

Hence, the two-point function has power-law behavior and the anomalous dimension of $\mathcal{O}$ vanishes for all values of $\lambda$ in the large-$N$ limit. The model based on the lagrangian (4.13) deformed by an additional mass term $i\mu\mathcal{O}$ has been studied in [27]. The two-point function (4.25), including normalization, is in agreement with the $\mu \to 0$ limit of the results of [27]. Remarkable, it has been shown in [27] that for $\mu = 0$ the scale invariance can be broken spontaneously at a particular critical value of the coupling[7] given by $\lambda_{\rm crit} = 1/4\pi$ (the coupling $\lambda'$ used in [27] is given by $\lambda' = 16\pi^2\lambda$). It would be interesting to find the bulk interpretation of this phenomenon.

[7]The scale invariance is broken by a dynamically generated fermion mass $\langle\mathcal{O}_2\rangle_{\lambda_{\rm crit}}$. For fixed $\mu \neq 0$, the two-point function has a pole in momentum space with mass proportional to $\sqrt{1-(\lambda/\lambda_{\rm crit})^2}\;\mu$ for $\lambda \sim \lambda_{\rm crit}$, while for fixed momentum the two-point function reduces to (4.25) in the limit $\mu \to 0$.



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The above analysis provides evidence for analytical dependence of $W_\lambda$ on $\lambda$. Note that while the deformation does not change the scaling dimension of $\mathcal{O}$, it breaks parity invariance of the theory. Once parity is restored at $\lambda = \infty$, the parity of the operator $\mathcal{O}$ has flipped from its value at $\lambda = 0$. Hence the scalar components of $\mathcal{O}$ have the following group theoretic content:

$$B_0 : \ D(1,0)_+ \oplus D(2,0)_- \,, \tag{4.26}$$

$$B_\infty : \ D(1,0)_- \oplus D(2,0)_+ \,. \tag{4.27}$$

The continuity in $\lambda$ also implies that the Legendre transformation (4.8) between the generating functionals at $B_0$ and $B_\infty$ can be reproduced on the CFT side by integrating out the Lagrange multiplier field $\Sigma$. To see this, following [20] we let $\Sigma \to \Sigma - J$ in (4.16) and take the limit $\lambda \to \infty$. Up to contact terms and overall normalization this yields:

$$e^{iW_\infty[\tilde{J}]} = \int D\Sigma e^{i\int \tilde{J}\Sigma + iW_0[\Sigma]} \,, \tag{4.28}$$

where $\tilde{J} = -J/\lambda$, which in turn yields the Legendre transformation (4.8) upon integrating out $\Sigma$ in the large $N$ limit.

Finally we propose a few tests of the correspondence at the level of scalar three-point amplitudes in the case of the $B_0$ and $B_\infty$ boundary conditions. The $B^3$ and $BA^2$ amplitudes and the corresponding correlators on the CFT side vanish trivially due to parity invariance. From the results in section 2.3 we know that the remaining cubic $A^3$ and $AB^2$ couplings in the HS gauge theory also vanish. Hence the associated bulk amplitudes, and the corresponding CFT correlators, vanish provided the integral over bulk-to-boundary propagators are finite. Hence, the $AB^2$ amplitude vanishes for $B_0$ boundary condition, in agreement with the fact that $\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_2\rangle_0 = 0$, while the $A^3$ amplitude vanishes for $B_\infty$ boundary conditions, which yields the prediction that $\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\rangle_\infty = 0$. This prediction is in agreement with the Legendre transformation rule (A.13) [21], and it would be interesting to verify it directly at strong coupling.

The $A^3$ amplitude subject to the $B_0$ boundary condition and the $AB^2$ amplitude subject to the $B_\infty$ boundary condition involve integrals over bulk-to-boundary propagators that require regularization. In dimensional regularization, for example, both of these integrals diverge as $(D-4)^{-1}$ in $D$ dimensions. The $A^3$ amplitude corresponds to the non-vanishing free-field theory correlator $\langle \mathcal{O}_1\mathcal{O}_1\mathcal{O}_1\rangle_0$, which implies that the three-point scalar self-coupling in the bulk theory is proportional to $D-4$ [21]. This in turn implies the $AB^2$ amplitude is finite as well, which leads to the prediction that the extremal correlator $\langle \mathcal{O}_2\mathcal{O}_1\mathcal{O}_1\rangle_\infty$ in the strongly coupled theory is non-vanishing. The above discussion is summarized in table 2.

### 4.2 The type-A/B model as AdS dual of $O(N)$ vector/gross-neveu model

As explained in section 3, the $\mathcal{N} = 1$ HS gauge theory field equations admit two different bosonic truncations, leading to the minimal bosonic type-A and type-B models which retain the scalar or the pseudo-scalar of the Wess-Zumino multiplet, respectively. Taking

| | | $A^3$ | $AB^2$ |
|---|---|---|---|
| $B_0$ | Bulk: | $g_3 \int K_1 K_1 K_1$ | $g_3 \int K_1 K_2 K_2$ |
| | CFT: | $\frac{1}{x_{12}x_{23}x_{31}}$ | $\frac{1}{x_{12}x_{13}} \delta^3(x_{23})$ |
| $B_\infty$ | Bulk: | $g_3 \int K_2 K_2 K_2$ | $g_3 \int K_2 K_1 K_1$ |
| | CFT: | $\delta^3(x_{12})\delta^3(x_{23})$ | $\frac{1}{x_{12}^2 x_{31}^2}$ |

**Table 2:** Schematic summary of $A^3$ and $AB^2$ amplitudes in $\mathrm{AdS}_4$ and their corresponding correlators on the CFT side. The bulk coupling $g_3$ vanishes in $D = 4$ dimensions and is proportional to $D - 4$ in $D$ dimensions. The quantities $K_\Delta$ ($\Delta = 1, 2$) are bulk-to-boundary propagators. The integrals $\int K_2K_2K_2$ and $\int K_1K_2K_2$ are finite while the integrals $\int K_2K_1K_1$ and $\int K_1K_1K_1$ are proportional to $(D - 4)^{-1}$. The correlators are related using the amputation formula (A.13), and their momentum space representations are given in the appendix.

also the $B_\lambda$ boundary conditions defined in (4.5) into account, the generating functional $W_\lambda[\Phi]$ can be consistently truncated in the cases of the $B_0$ and $B_\infty$ boundary conditions. The consistent truncation can also be applied to the Legendre transformation formula (4.8). This yields two pairs of bosonic generating functionals related by Legendre transformations:

$$\text{Type A}: \quad W_\infty[\alpha_+] = W_0[\beta_+] + \int d^3x \beta_+ \alpha_+ \,, \tag{4.29}$$

$$\text{Type B}: \quad W_\infty[\beta_-] = W_0[\alpha_-] + \int d^3x \alpha_- \beta_- \,. \tag{4.30}$$

Turning to their holographic duals, the $hs(4)$ invariant generating functional $W_0[\beta_+]$ corresponds to a free $O(N)$ vector model [7]. Here we propose that the $hs(4)$ invariant generating functional $W_0[\alpha_-]$ corresponds to the free $O(N)$ fermion model. Indeed the spectra of massless fields of the type-A and -B models given in (3.6)–(3.7) are in one-to-one correspondence with the bilinear $O(N)$ invariant operators in the corresponding two free field theories. In the case of $\mathcal{N} = 1$, the bulk scalars $A$ and $B$ couple to $\varphi^2$ and $\psi^2$, respectively, which clearly is consistent with the type-A/B truncations at the level of the scalar multiplet.

For the higher spin fields the truncation acts slightly differently. In the spin $s = 2$ sector in the $\mathcal{N} = 1$ theory, the graviton arises from the gauging of the $\mathrm{SO}(3, 2)$ generators $M_{AB}$, and a second spin $s = 2$ field arises from the gauging of $\widetilde{M}_{AB} = M_{AB}\Gamma$. These fields couple to $T = T(\varphi) + T(\psi)$ and $\widetilde{T} = T(\varphi) - T(\psi)$, respectively, where $T(\varphi)$ and $T(\psi)$ are the stress-energy tensors for the free scalars and fermions, respectively, and $T$ is the total stress-energy tensor of the free $\mathcal{N} = 1$ supersymmetric $O(N)$ vector model. From (3.1) it follows that the graviton in the type-A and -B model arises from the gauging of $(M_{AB} + \widetilde{M}_{AB})/2$ and $(M_{AB} - \widetilde{M}_{AB})/2$, respectively, and hence couples to $(T + \widetilde{T})/2 = T(\varphi)$ and $(T - \widetilde{T})/2 = T(\psi)$, respectively.

The Legendre transformations (4.29) and (4.30) are realized on the CFT side by adding double-trace deformations to the free CFTs and taking strong coupling limits [20, 8] (these deformations are not truncations of the $\mathcal{N} = 1$ supersymmetric double-trace deformation used to realize (4.8) in the $\mathcal{N} = 1$ supersymmetric $O(N)$ vector model). In the case of the type-A model, this leads to the Klebanov-Polyakov conjecture, according to which the generating functional $W_\infty[\alpha_+]$ is the AdS dual of the strongly coupled IR fixed point of the relevant $\frac{\lambda}{2N}\int d^3x(\varphi^2)^2$ deformation of the $O(N)$ vector model. Note that this duality is conjectured to hold *in the strict* $N \to \infty$ *limit.*

From the results of section 2.3 it follows that the $A^3$ amplitude in the type-A model vanishes in the case of $D(2,0)_+$ boundary condition. This is in agreement with the vanishing of the three-point scalar correlator in the strongly coupled $O(N)$ model [23], and therefore provides a non-trivial test of the Klebanov-Polyakov conjecture. In the case of the $D(1,0)_+$ boundary condition, the $A^3$ amplitude needs to be regularized as discussed in the previous section.

In the case of the type-B model, we conjecture that $W_\infty[\beta_-]$ is the AdS dual of the strongly coupled UV fixed point of the three-dimensional Gross-Neveu model defined by the four-fermion interaction $\frac{\lambda}{2N}\int d^3(\psi^2)^2$. Though this irrelevant double-trace deformation is non-renormalizable by the usual power-counting argument, it is known to be renormalizable in the $1/N$ expansion, and drives the theory to a strongly coupled fixed point in the UV where $\Delta((\psi)^2) = 1$.

The vanishing of the $B^3$ coupling in the type-B model and the corresponding correlator in the Gross-Neveu model follows from parity invariance, and therefore does not provide a non-trivial test of the above proposal. A non-trivial test would be the matching of the graviton-$B^2$ amplitude with the corresponding CFT correlator. We shall comment on this bulk coupling in section 5 in the context of the perturbative stability of the HS theory.

## 5. Summary and discussion

We have described a parity invariant minimal $\mathcal{N} = 1$ model and its bosonic truncations, namely the type-A and -B minimal bosonic HS theories. Both bosonic models have local $hs(4)$ symmetry and spectrum consisting of massless fields with spin $s = 0, 2, 4, \ldots$ each occurring once. The scalar is even under parity in the type-A model and odd in the type-B model.

In the case of HS invariant boundary conditions the above models correspond to free field theories on the boundary of AdS [7]. The models also admit boundary conditions which break HS symmetry, and which are related to the HS invariant boundary conditions by Legendre transformations [8]. On the field theory side, the Legendre transformations correspond to strong coupling limits of various double-trace deformations [17 − 20]. The various deformations and boundary conditions are summarized in table 3.

We have examined certain couplings in the minimal $\mathcal{N} = 1$ HS theory in four dimensions and in particular found that the quadratic scalar contributions to the scalar field equation vanish. By truncation, the same holds in the type-A and -B models. In the case of the type-B model, the $B^3$ amplitudes and the corresponding CFT correlators vanish trivially for both $D(1,0)_-$ and $D(2,0)_-$ boundary conditions by parity invariance. In the case of the type-A model, the $A^3$ amplitude vanishes for the $D(2,0)_+$ boundary condition, in agreement with the result for the strongly coupled $O(N)$ vector model obtained by Petkou [23]. In the case of the $\mathcal{N} = 1$ model, the predictions for the strongly coupled fixed point based on analyzing cubic scalar amplitudes are given in table 2.



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| Model | Deformation | $\lambda = 0$ | $\lambda = \infty$ |
|---|---|---|---|
| $\mathcal{N} = 1$ | $\frac{\lambda}{2} \int d^2\theta d^3x (\mathrm{tr} W^2)^2$ | $D(1,0)_+ \oplus D(2,0)_-$ | $D(1,0)_- \oplus D(2,0)_+$ |
| Type A | $\frac{\lambda}{2} \int d^3x (\mathrm{tr}\varphi^2)^2$ | $D(1,0)_+$ (UV) | $D(2,0)_+$ (IR) |
| Type B | $\frac{\lambda}{2} \int d^3x (\mathrm{tr}\psi^2)^2$ | $D(2,0)_-$ (IR) | $D(1,0)_-$ (UV) |

**Table 3:** The field theory deformations and corresponding boundary conditions at the free fixed point at $\lambda = 0$ and the strongly coupled fixed point at $\lambda = \infty$. In the case of $\mathcal{N} = 1$ the two fixed points are connected by a line of fixed points. In the case of the type-A and -B models the fixed points are related by RG flows.

Another test of the correspondence is to construct domain wall solutions to the type-A and type-B field equations (3.3)–(3.4). The domain wall should have the topology of AdS spacetime and break $hs(4)$ down to a possibly infinite dimensional subalgebra with maximal finite subalgebra ISO(2,1). Moreover, it should interpolate between an asymptotic AdS region close to the boundary (UV) and another one in the deep interior (IR), such that $D(1,0)$ scalar fluctuations in the UV interpolate into $D(2,0)$ fluctuations in the IR.

In this paper we have been mainly concerned with the HS/CFT correspondence in the leading order in the $1/N$-expansion. There are several issues related to the $1/N$-corrections. To begin with, on the field theory side the $1/N$-corrections require $\lambda \neq 0$. In particular, these corrections show up as anomalies in the HS current conservation laws. This corresponds to spontaneous breaking of the HS gauge symmetry in the bulk. In general, the Goldstone modes in the bulk couple to the anomaly operators on the field theory side [7]. In the case of double-trace deformations, the anomaly of the spin $s$ current has a double-trace character of a particular form suggesting that the candidate Goldstone mode for the corresponding massless field is a composite state formed out of the scalar field and a massless spin $s-2$ field [22]. In general, Higgsing of a massless spin $s$ field with parity $(-1)^s$ in the $D(s+1,s)_{(-1)^s}$ representation requires a Goldstone mode in the $D(s+2,s-1)_{(-1)^{s-1}}$ representation [22]. Stated group theoretically, $D(s+1+\gamma,s)$ is irreducible for $\gamma > 0$ while it decomposes into a massless and a massive irrep in the limit when the anomalous dimension $\gamma \to 0$:

$$\lim_{\gamma\to 0} D(s+1+\gamma,s)_{(-1)^s} = D(s+1,s)_{(-1)^s} \oplus D(s+2,s-1)_{(-1)^{s-1}}\,, \qquad s \geq 1\,. \tag{5.1}$$

In the case of the type-A and -B models, the Goldstone mode for spin $s \geq 4$ is contained in $D(2,0)_+ \otimes D(s-1,s-2)_{(-1)^s}$ and $D(1,0)_- \otimes D(s-1,s-2)_{(-1)^s}$, as follows from

$$D(\Delta,0)_\epsilon \otimes D(s-1,s-2)_{(-1)^s} = \sum_{j=0}^{\infty}\sum_{l=1}^{\infty} D(\Delta+s-1+l+j, s-2+j)_{\epsilon(-1)^{s+j+l}}\,, \qquad s \geq 4\,. \tag{5.2}$$

Moreover, there is no candidate Goldstone mode for the graviton, as expected. In the case of the $\mathcal{N} = 1$ model, the Goldstone modes arise in a similar fashion. Thus, in all HS models studied here, candidate Goldstone modes arise for the boundary conditions corresponding to the $\lambda = \infty$ limit on the field theory side.

The fact that the Goldstone modes are composite states suggests that the actual breaking mechanism involves radiative corrections to the bulk theory. This raises the question



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whether the HS gauge theories provide self-contained and consistent quantum theories of gravity. This is by no means clear and requires further study.

It is natural to embed the free $O(N)$ vector models into free singleton matrix models, and consider corresponding AdS duals with massless as well as massive fields, some of which are candidate fundamental Goldstone modes [7]. The free matrix theory has a more intricate $1/N$ expansion than the free $O(N)$ theory [3], due to mixing between single-trace and multi-trace operators. However, the corresponding bulk action appears to admit a consistent truncation to the massless sector, which is problematic in attempting to interpret it as being an effective action for a quantum theory. This suggests that loop-corrections require the $\lambda = \infty$ boundary conditions associated with the strongly coupled CFT, though there still remains to resolve the issue of whether bulk loops are well-defined in the massless theory with $O(N)$ dual, or if massive fields are required with dual matrix description. One possibility is that the full quantum theory requires massive fields, in which case the embedding of the whole scenario into string theory would be natural [7, 25, 26], and the effective action for the massless fields can be obtained by integrating out the massive fields once the HS gauge symmetry has been broken.

Clearly, the stability of the bulk theory at the quantum level is a highly non-trivial issue, especially given the fact that the positivity of the total energy is not built into the HS field equations. In order for the theory to make sense perturbatively it should be stable when expanded around its AdS vacuum. As a first step it would be desirable to quadratic contributions to the stress-energy tensor. Progress in this direction has been made in [28] where the quadratic contributions from the scalar field have been extracted from the full field equations of the type-A and B models. This result generalizes straightforwardly to the minimal $\mathcal{N} = 1$ model as follows:

$$T_{\mu\nu} = \tau_{\mu\nu}(\phi) + \text{h.c.}\,, \tag{5.3}$$

$$\tau_{\mu\nu}(\phi) = -\frac{4}{9} g_{\mu\nu}\phi^2 + \sum_{k=0}^{\infty}\Big( a_k g_{\mu\nu}(\phi_{\rho_1\ldots\rho_{k+1}})^2 + {} + b_k \phi_{\mu\rho_1\ldots\rho_k}\phi_\nu{}^{\rho_1\rho_k} + c_k \phi_{\mu\nu\rho_1\ldots\rho_k}\phi^{\rho_1\ldots\rho_k}\Big)\,, \tag{5.4}$$

where $\phi_{\mu_1\ldots\mu_n} = \nabla_{(\mu_1}\cdots\nabla_{\mu_n)}\phi -$ traces, and $a_k$, $b_k$ and $c_k$ are numerical coefficients which are given in [28]. The stress-energy tensors in the type-A and -B models are obtained by substituting $\phi = A + iB$, which gives

$$\tau_{\mu\nu}(\phi) + \text{h.c.} = \tau_{\mu\nu}(A) - \tau_{\mu\nu}(B)\,, \tag{5.5}$$

and retaining $A$ or $B$. The relative sign in (5.5) is surprising and intriguing, both from the point of view of $\mathcal{N} = 1$ supersymmetry and that of stability (though the sign matches the structure of the supersymmetric Legendre transformation formula as explained below (4.8)). By construction, the field equations are invariant under local $hs(1|4)$ symmetries, including $\mathcal{N} = 1$ supersymmetry, whose explicit form follows from the basic integrable constraints giving rise to the physical field equations [9]. In the view of the above result, it would be instructive to examine exactly how supersymmetry is realized on-shell up to the second order in the weak field expansion.

In examining the stability of the scalar fluctuations, one has to take into account the fact that all terms in $\tau_{\mu\nu}$ are of the same order, since the coefficients in (5.4) behave as $2^k/(k!)^2$ for large $k$, while $\nabla^\mu\phi_{\mu\mu_1\ldots\mu_k} \sim k^2\phi_{\mu_1\ldots\mu_k}$. It may be possible to make stability manifest by absorbing all higher derivative terms into a field redefinition, though the feasibility of such a field redefinition remains to be seen. On the other hand, it may also be the case that $\tau_{\mu\nu}$ is positive only for certain boundary condition on the scalar field (unlike the canonical stress-tensor which is positive for both boundary conditions). Since loop-corrections to the HS theory only appear to make sense when the holographic dual is strongly coupled, a natural outcome of the stability analysis would be that $(-1)^{\Delta_\pm}\tau_{\mu\nu}$ is positive.

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## A. Conventions and useful formulae

In this appendix we summarize our conventions for $\mathcal{N}=1$, $d=3$ superspace and a few other formula that we use in section 4.1. We work in lorentzian signature, $\eta_{ab}=(-++)$, and with two-component Majorana spinors.[8] The Dirac matrices $(\sigma^a)_{\alpha\beta}$ are real and symmetric and obey

$$(\sigma_a)_\alpha{}^\beta(\sigma_b)_{\beta\gamma} = \epsilon_{\alpha\gamma}\eta_{ab} + \epsilon_{abc}(\sigma^c)_{\alpha\gamma}\,. \tag{A.1}$$

We use the north-west-south-east convention, $\theta^\alpha=\epsilon^{\alpha\beta}\theta_\beta$ and $\theta_\alpha=\theta^\beta\epsilon_{\beta\alpha}$, and spinor bilinears are written as $\bar\theta\eta=\theta^\alpha\eta_\alpha$ and $\bar\theta\sigma^a\eta=\theta^\alpha(\sigma^a)_\alpha{}^\beta\eta_\beta$. The supercovariant derivative is defined by

$$D_\alpha = \partial_\alpha + i(\sigma^\mu)_{\alpha\beta}\theta^\beta\partial_\mu\,. \tag{A.2}$$

The integration over odd superspace coordinates is defined by

$$\int d^2\theta\delta^2(\theta) = 1\,, \tag{A.3}$$

where the integration measure and the $\delta$-function are defined by

$$d^2\theta = \frac{d\bar\theta d\theta}{2i}\,, \qquad \delta^2(\theta) = \frac{\bar\theta\theta}{2i}\,. \tag{A.4}$$

The superspace $\delta$-function has the following representation

$$\delta^2(\theta)\delta^3(x) = \int\frac{dP}{(2\pi)^3}e^{iPZ}\,, \tag{A.5}$$

[8] The computation in section 4.1 may equally well be carried out in euclidean space using the conventions of [29].

where

$$dP = d^2\pi d^3p\,, \qquad PZ = p^a x_a - i\pi^\alpha\theta_\alpha\,. \tag{A.6}$$

Fourier transformation in superspace is defined by

$$F(Z) = \int \frac{dP}{(2\pi)^{3/2}} e^{iPZ}\widetilde{F}(P)\,, \tag{A.7}$$

$$\widetilde{F}(P) = \int \frac{dZ}{(2\pi)^{3/2}} e^{-iPZ}F(Z)\,. \tag{A.8}$$

The operator $\mathcal{O}$ defined in (4.10) has the following two-point function in the free theory

$$\left\langle \mathcal{O}(Z)\mathcal{O}(Z')\right\rangle_0 = G(Z,Z') = G(Z-Z') = \frac{1}{(x-x'+i\bar\theta\sigma\theta')^2}\,, \tag{A.9}$$

where we have defined

$$Z - Z' = (x - x' + i\bar\theta\sigma\theta', \theta - \theta')\,. \tag{A.10}$$

The momentum space representation of the two-point function is given by

$$G(P) = \delta^2(\pi)G(p)\,, \qquad G(p) = \int \frac{d^3x}{(2\pi)^{3/2}}\frac{e^{-ipx}}{x^2} = \frac{(2\pi)^{3/2}}{4\pi\sqrt{-p^2}}\,. \tag{A.11}$$

At the level of three-point functions the Legendre transformation amounts to amputation, which takes the following form in the case of scalar operators:

$$\begin{aligned}
&\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\rangle_\infty = \\
&\quad = \int d^3x_1' d^3x_2' d^3x_2' G^{-1}_{\Delta_1}(x_1,x_1')G^{-1}_{\Delta_2}(x_2,x_2')G^{-1}_{\Delta_3}(x_3,x_3') \times \\
&\qquad \times \left\langle \mathcal{O}_{\tilde\Delta_1}(x_1')\mathcal{O}_{\tilde\Delta_2}(x_2')\mathcal{O}_{\tilde\Delta_3}(x_3')\right\rangle_0\,,
\end{aligned} \tag{A.12}$$

where $G_\Delta(x,y) = \langle\mathcal{O}_\Delta(x)\mathcal{O}_\Delta(y)\rangle_0$ and $\tilde\Delta = 3 - \Delta$. The amputation becomes local in momentum space

$$\langle \mathcal{O}_{\Delta_1}(p_1)\mathcal{O}_{\Delta_2}(p_2)\mathcal{O}_{\Delta_3}(p_3)\rangle_\infty = G^{-1}_{\Delta_1}(p_1)G^{-1}_{\Delta_2}(p_2)G^{-1}_{\Delta_3}(p_3)\left\langle \mathcal{O}_{\tilde\Delta_1}(p_1)\mathcal{O}_{\tilde\Delta_2}(p_2)\mathcal{O}_{\tilde\Delta_3}(p_3)\right\rangle_0\,. \tag{A.13}$$

The momentum space representation of the non-vanishing three-point functions of the scalar components of the operator $\mathcal{O}$ are given at weak and strong coupling by:

$$\langle \mathcal{O}_1(p_1)\mathcal{O}_1(p_2)\mathcal{O}_1(p_3)\rangle_0 \sim \frac{\delta^3(p_1+p_2+p_3)}{p_1p_2p_3}\,, \tag{A.14}$$

$$\langle \mathcal{O}_1(p_1)\mathcal{O}_2(p_2)\mathcal{O}_2(p_3)\rangle_0 \sim \frac{\delta^3(p_1+p_2+p_3)}{p_1}\,, \tag{A.15}$$

$$\langle \mathcal{O}_2(p_1)\mathcal{O}_1(p_2)\mathcal{O}_1(p_3)\rangle_\infty \sim \frac{\delta^3(p_1+p_2+p_3)}{p_2p_3}\,, \tag{A.16}$$

$$\langle \mathcal{O}_2(p_1)\mathcal{O}_2(p_2)\mathcal{O}_2(p_3)\rangle_\infty \sim \delta^3(p_1+p_2+p_3)\,. \tag{A.17}$$



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